# All's Well That Ends Well: Supplementary Proofs

This document complements the paper "All's Well That Ends Well: Guaranteed Resolution of Simultaneous Rigid Body Impact" and provides detailed proofs of several claims therein: that pairwise Gauss-Seidel-like algorithms and Generalized Reflections, when modified according to the template shown in Algorithm 3, satisfy all the inexact impact operator desiderata, and hence are guaranteed to terminate, just as are their exact arithmetic counterparts.

## A DETAILED INEXACT ARITHMETIC PROOFS

Here we will prove the claims in §7: that both the inexact pairwise Gauss-Seidel method described in Algorithm 3, as well as the Smith et al.'s Generalized Reflections algorithm [2012], satisfy the inexact impact operator axioms ($\epsilon$NORM)–($\epsilon$MOD). We will assume the following computation model: real numbers are approximated using floating-point arithmetic, with machine epsilon $\varepsilon < 1$ and minimum representable magnitude $\eta < \varepsilon$. We assume that no intermediate calculation overflows; we then have an associated rounding operator $\mathrm{fl}[x]$, so that for every exact quantity $x$,

$$x - |x|\varepsilon - \eta \leq \mathrm{fl}[x] \leq x + |x|\varepsilon + \eta.$$

For calculations we will make use of the weaker, more convenient bound

$$x - |x|\varepsilon - \varepsilon \leq \mathrm{fl}[x] \leq x + |x|\varepsilon + \varepsilon.$$

Arithmetic operations and square roots are assumed to take place in infinite precision, and then rounded; we will write $\mathrm{fl}[E]$ to denote that every operation in the expression $E$ is performed in this way, e.g. $\mathrm{fl}[x+y] = \mathrm{fl}[\mathrm{fl}[x] + \mathrm{fl}[y]]$. Finally, we will assume that $\dot{\mathbf{q}}_i$ and small integer constants are represented exactly, but that $M$, $M^{-1}$, and $N$ must be rounded.

If $\varepsilon$ is too large, the properties ($\epsilon$NORM), ($\epsilon$DRIFT), and ($\epsilon$MOD) cannot be guaranteed. We will prove that both pairwise Gauss-Seidel and Generalized Reflections satisfy these properties for $\varepsilon$ sufficiently small, and give a constructive bound for $\varepsilon$ in terms of the magnitudes of input quantities like $\dot{\mathbf{q}}_0$, $M$, $N$, etc. For both algorithms, we will first look at drift, and construct a $C$ which is used in the definition of ($\epsilon$DRIFT) as a certificate that energy cannot grow unbounded over the course of several iterations. The proof of no drift will already impose a bound on $\varepsilon$; intuitively, if the machine precision is too large, the renormalization of the velocity after every iteration in Algorithms 3 and 4 itself introduces so much error into the computation of $\dot{\mathbf{q}}_{i+1}$ that despite the renomalization, its magnitude cannot be bounded.

Once we have constructed a $C$, we also need an $\epsilon$. We will show that ($\epsilon$NORM) imposes a lower bound of $\epsilon$, and that this lower bound decreases to zero as $\varepsilon$ decreases. We end by proving ($\epsilon$MOD) hold, provided that $\epsilon$ is not too large. The upper bound is constant, and the lower bound shrinks as $\varepsilon$ shrinks, so that it is always possible to find an $\epsilon$ if $\varepsilon$ is sufficiently small.

### A.1 Pairwise Gauss-Seidel

In this section, we derive an $\epsilon$ and $C$ for which the modified pairwise GS algorithm described in section 7 satisfies the six criteria ($\epsilon$NORM)–($\epsilon$MOD). Three of these, ($\epsilon$KIN), (ONE) and ($\epsilon$VIO), are obvious from the construction of the algorithm. We first prove ($\epsilon$DRIFT) by induction on the iteration $i$: suppose it holds for the first $i$ iterations of Algorithm 3. Then

$$\begin{aligned}
\frac{1}{2}\|\dot{\mathbf{q}}_i\|_M^2 &\leq \frac{1}{2}\|\dot{\mathbf{q}}_0\|_M^2 + C \\
\|\dot{\mathbf{q}}_i\|_M^2 &\leq \|\dot{\mathbf{q}}_0\|_M^2 + 2C \\
\lambda_{\min}\|\dot{\mathbf{q}}_i\|_2^2 &\leq \lambda_{\max}\|\dot{\mathbf{q}}_0\|_2^2 + 2C \\
\|\dot{\mathbf{q}}_i\|_2 &\leq \alpha_1 + \beta_1\sqrt{C},
\end{aligned}$$

for

$$\begin{aligned}
\alpha_1 &= \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\|\dot{\mathbf{q}}_0\|_2 \\
\beta_1 &= \sqrt{2}.
\end{aligned}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalue of $M$, respectively. Since $\|\dot{\mathbf{q}}_i\|_\infty \leq \|\dot{\mathbf{q}}_i\|_2$ we also have that

$$\|\dot{\mathbf{q}}_i\|_\infty \leq \alpha_1 + \beta_1\sqrt{C}.$$

We now bound $\bar{\mathbf{p}} = \mathrm{fl}[\dot{\mathbf{q}}_i - 2\langle\dot{\mathbf{q}}_i, \mathbf{n}\rangle M^{-1}\mathbf{n}]$, where $\mathbf{n}$ is some constraint gradient selected by Algorithm 3. The following fact will be useful: for a sequence of numbers $x_1, \ldots, x_d$, it can be shown by induction on $d$ that

$$\left|\mathrm{fl}\left[\sum_{j=1}^{d} x_i\right] - \sum_{j=1}^{d}\mathrm{fl}[x_i]\right| \leq \left(d + \sum_{j=1}^{d}|\mathrm{fl}[x_i]|\right)\varepsilon(1+\varepsilon)^{d-1}.$$

We now proceed to bound $\bar{\mathbf{p}}$. First,

$$\left|\mathrm{fl}[\dot{\mathbf{q}}_i^j\mathbf{n}^j] - \dot{\mathbf{q}}_i^j\mathbf{n}^j\right| \leq \left(\left|\dot{\mathbf{q}}_i^j\right|\left|\mathrm{fl}[\mathbf{n}^j]\right| + 1\right)\varepsilon$$

where $\mathbf{n}^j$ denotes the $j$th coordinates of the vector $\mathbf{n}$. We can write these bounds as

$$\left|\mathrm{fl}[\dot{\mathbf{q}}_i^j\mathbf{n}^j] - \dot{\mathbf{q}}_i^j\mathbf{n}^j\right| \leq \varepsilon\left(\alpha_2 + \beta_2\sqrt{C}\right)$$

where

$$\begin{aligned}
\alpha_2 &= \alpha_1(\|\mathbf{n}\|_\infty(1+\varepsilon) + \varepsilon) + 1 \\
\beta_2 &= \beta_1(\|\mathbf{n}\|_\infty(1+\varepsilon) + \varepsilon).
\end{aligned}$$

Since

$$\left|\dot{\mathbf{q}}_i^j\mathbf{n}^j\right| \leq \|\mathbf{q}_i\|_\infty\|\mathbf{n}\|_\infty \leq \|\mathbf{n}\|_\infty(\alpha_1 + \beta_1\sqrt{C}),$$

summing over $j$ gives

$$\left|\mathrm{fl}\left[\langle\dot{\mathbf{q}}_i, \mathbf{n}\rangle\right] - \langle\dot{\mathbf{q}}_i, \mathbf{n}\rangle\right| \leq \varepsilon\left(\alpha_3 + \beta_3\sqrt{C}\right)$$

where

$$\begin{aligned}
\alpha_3 &= (1 + \|\mathbf{n}\|\alpha_1 + 2\varepsilon\alpha_2)d(1+\varepsilon)^{d-1} \\
\beta_3 &= (\|\mathbf{n}\|\beta_1 + 2\varepsilon\beta_2)d(1+\varepsilon)^{d-1}.
\end{aligned}$$

Switching gears,

$$\left|\mathrm{fl}\left[\left(M^{-1}\right)^{kj}\right]-\left(M^{-1}\right)^{kj}\right| \leq \left\|M^{-1}\right\|_\infty \varepsilon+\varepsilon$$

and

$$\left|\mathrm{fl}\left[\mathbf{n}^j\right]-\mathbf{n}^j\right| \leq (\|\mathbf{n}\|_\infty+1)\varepsilon,$$

so that

$$\left|\mathrm{fl}\left[\left(M^{-1}\right)^{kj}\mathbf{n}^j\right]-\left(M^{-1}\right)^{kj}\mathbf{n}^j\right| \leq \varepsilon\alpha_4$$

where

$$\alpha_4 = 7\left\|M^{-1}\right\|_\infty\|\mathbf{n}\|_\infty+4\left\|M^{-1}\right\|_\infty+4\|\mathbf{n}\|_\infty+3.$$

Summing again over $j$ we can bound

$$\left|\mathrm{fl}\left[\left(M^{-1}\mathbf{n}\right)^k\right]-\left(M^{-1}\mathbf{n}\right)^k\right| \leq \varepsilon\alpha_5$$

where

$$\alpha_5 = \left(1+\left\|M^{-1}\right\|_\infty\|\mathbf{n}\|_\infty+2\varepsilon\alpha_4\right)(1+\varepsilon)^{d-1}.$$

Now since

$$\left|\mathrm{fl}\left[\left(M^{-1}\mathbf{n}\right)^k\right]\right| \leq d\left\|M^{-1}\right\|_\infty\|\mathbf{n}\|_\infty+\varepsilon\alpha_5$$

we have that

$$\left|\mathrm{fl}\left[\langle\dot{\mathbf{q}}_i,\mathbf{n}\rangle\left(M^{-1}\mathbf{n}\right)^j\right]-\langle\dot{\mathbf{q}}_i,\mathbf{n}\rangle\left(M^{-1}\mathbf{n}\right)^j\right| \leq \varepsilon(\alpha_6+\beta_6\sqrt{C})$$

for

$$\begin{aligned}\alpha_6 &= 1+(1+2\alpha_3)d\left\|M^{-1}\right\|_\infty\|\mathbf{n}\|_\infty+2\alpha_5 d\|\mathbf{n}\|_\infty\alpha_1+2\alpha_3\alpha_5\\ \beta_6 &= 2\beta_3 d\left\|M^{-1}\right\|_\infty\|\mathbf{n}\|_\infty+2\alpha_5 d\|\mathbf{n}\|_\infty\beta_1+2\alpha_5\beta_3,\end{aligned}$$

where we have made liberal use of the fact that $\varepsilon^2<\varepsilon$ to simplify the above expressions. Then

$$\begin{aligned}&\left|\mathrm{fl}\left[-2\langle\dot{\mathbf{q}}_i,\mathbf{n}\rangle\left(M^{-1}\mathbf{n}\right)^j\right]+2\langle\dot{\mathbf{q}}_i,\mathbf{n}\rangle\left(M^{-1}\mathbf{n}\right)^j\right|\\ \leq&\varepsilon(\alpha_7+\beta_7\sqrt{C})\end{aligned}$$

where

$$\begin{aligned}\alpha_7 &= 1+4\alpha_6+2d\left\|M^{-1}\right\|_\infty\|\mathbf{n}\|_\infty^2\alpha_1\\ \beta_7 &= 4\beta_6+2d\left\|M^{-1}\right\|_\infty\|\mathbf{n}\|_\infty^2\beta_1.\end{aligned}$$

Finally, we bound $\bar{\mathbf{p}}$ in terms of $\mathbf{p}=\dot{\mathbf{q}}_i-2\langle\dot{\mathbf{q}}_i,\mathbf{n}\rangle M^{-1}\mathbf{n}$. We have that

$$\left|\bar{\mathbf{p}}^j-\mathbf{p}^j\right| \leq \varepsilon(\alpha_8+\beta_8\sqrt{C}) \tag{4}$$

for

$$\begin{aligned}\alpha_8 &= 1+\alpha_1+2\alpha_7+2d\left\|M^{-1}\right\|_\infty\|\mathbf{n}\|_\infty^2\alpha_1\\ \beta_8 &= \beta_1+2\beta_7+2d\left\|M^{-1}\right\|_\infty\|\mathbf{n}\|_\infty^2\beta_1.\end{aligned}$$

Next, we need to bound the norm $\mathrm{fl}\left[\|\bar{\mathbf{p}}\|_M\right]$ in the denominator of the coefficient of the velocity update step. We can use the fact that

$$\left|\mathbf{p}^j\right| \leq \|\mathbf{p}\|_\infty \leq \frac{\|\mathbf{p}\|_M}{\sqrt{\lambda_{\min}}} = \frac{\|\dot{\mathbf{q}}_i\|_M}{\sqrt{\lambda_{\min}}} \leq \frac{\|\dot{\mathbf{q}}_0\|_M+\sqrt{2C}}{\sqrt{\lambda_{\min}}}$$

to get

$$\left|\mathrm{fl}\left[M^{kj}\bar{\mathbf{p}}^j\right]-M^{kj}\mathbf{p}^j\right| \leq \varepsilon(\alpha_9+\beta_9\sqrt{C})$$

for

$$\begin{aligned}\alpha_9 &= 1+(2+3\|M\|_\infty)\left(\alpha_8+\frac{\|\dot{\mathbf{q}}_0\|_M}{\sqrt{\lambda_{\min}}}\right)\\ \beta_9 &= (2+3\|M\|_\infty)\left(\beta_8+\sqrt{\frac{2}{\lambda_{\min}}}\right).\end{aligned}$$

The summation formula then gives

$$\left|\mathrm{fl}\left[(M\bar{\mathbf{p}})^k\right]-(M\mathbf{p})^k\right| \leq \varepsilon(\alpha_{10}+\beta_{10}\sqrt{C})$$

where

$$\begin{aligned}\alpha_{10} &= d\alpha_9+\left(1+d\|M\|_\infty\frac{\|\dot{\mathbf{q}}_0\|_M}{\sqrt{\lambda_{\min}}}+\varepsilon\alpha_9\right)d(1+\varepsilon)^{d-1}\\ \beta_{10} &= d\beta_9+\left(d\|M\|_\infty\sqrt{\frac{2}{\lambda_{\min}}}+\varepsilon\beta_9\right)d(1+\varepsilon)^{d-1}\end{aligned}$$

Next, combining the last several bounds,

$$\left|\mathrm{fl}\left[\bar{\mathbf{p}}^j\,(M\bar{\mathbf{p}})^j\right]-\mathbf{p}^j\,(M\mathbf{p})^j\right| \leq \varepsilon(\alpha_{11}+\beta_{11}\sqrt{C}+\gamma_{11}C)$$

for

$$\begin{aligned}\alpha_{11} =& 1+\frac{d\|M\|_\infty\|\dot{\mathbf{q}}_0\|_M^2}{\lambda_{\min}}+2\alpha_8\alpha_{10}\\ &+2(\alpha_{10}+\alpha_8 d\|M\|_\infty)\frac{\|\dot{\mathbf{q}}_0\|_M}{\sqrt{\lambda_{\min}}}\\ \beta_{11} =& 2d\|M\|_\infty\frac{\|\dot{\mathbf{q}}_0\|_M\sqrt{2}}{\lambda_{\min}}+2\alpha_8\beta_{10}+2\alpha_{10}\beta_8\\ &+2(\beta_{10}+\beta_8 d\|M\|_\infty)\frac{\|\dot{\mathbf{q}}_0\|_M}{\sqrt{\lambda_{\min}}}\\ &+2(\alpha_{10}+\alpha_8 d\|M\|_\infty)\sqrt{\frac{2}{\lambda_{\min}}}\\ \gamma_{11} =& \frac{2d\|M\|_\infty}{\lambda_{\min}}+2\beta_8\beta_{10}+2(\beta_{10}+\beta_8 d\|M\|_\infty)\sqrt{\frac{2}{\lambda_{\min}}}.\end{aligned}$$

We apply the summation formula a second time to get the squared norm,

$$\left|\mathrm{fl}\left[\bar{\mathbf{p}}^T M\bar{\mathbf{p}}\right]-\mathbf{p}^T M\mathbf{p}\right| \leq \varepsilon\left(\alpha_{12}+\beta_{12}\sqrt{C}+\gamma_{12}C\right),$$

for

$$\begin{aligned}\alpha_{12} &= d\alpha_{11}+(1+d\|\dot{\mathbf{q}}_0\|_M+d\varepsilon\alpha_{11})d(1+\varepsilon)^{d-1}\\ \beta_{12} &= d\beta_{11}+(d\sqrt{2}+d\varepsilon\beta_{11})d(1+\varepsilon)^{d-1}\\ \gamma_{12} &= d\gamma_{11}+d^2\varepsilon\gamma_{11}(1+\varepsilon)^{d-1}.\end{aligned}$$

We can rewrite this bound in more convenient form, by completing the square, in anticipation of taking the square root:

$$\begin{aligned}&\left|\mathrm{fl}\left[\bar{\mathbf{p}}^T M\bar{\mathbf{p}}\right]-\mathbf{p}^T M\mathbf{p}\right|\\ \leq&\varepsilon\left(\frac{\beta_{12}}{2\sqrt{\gamma_{12}}}+\sqrt{\gamma_{12}}\sqrt{C}\right)^2+\varepsilon\left(\alpha_{12}-\frac{\beta_{12}^2}{4\gamma_{12}}\right).\end{aligned}$$

Finally, we have a bound on the norm of $\mathbf{p}$:

$$\left|\mathrm{fl}\left[\|\bar{\mathbf{p}}\|_M\right]-\|\mathbf{p}\|_M\right| \leq \varepsilon(\alpha_{13}+\beta_{13}\sqrt{C}), \tag{5}$$

where

$$\alpha_{13} = 1 + \|\dot{\mathbf{q}}_0\|_M + \frac{1+\varepsilon}{\sqrt{\varepsilon}}\left(\sqrt{\left|\alpha_{12} - \frac{\beta_{12}^2}{4\gamma_{12}}\right|} + \frac{\beta_{12}}{2\sqrt{\gamma_{12}}}\right)$$

$$\beta_{13} = \frac{1+\varepsilon}{\sqrt{\varepsilon}}\sqrt{\gamma_{12}}.$$

Notice that since $\|M\|_\infty \geq \lambda_{\min}$, $\gamma_{12} > 1$ and so the denominators in $\alpha_{13}$ are bounded well away from zero.

The last piece we need for computing $\dot{\mathbf{q}}_{i+1}$ is the norm of the initial velocity, $\|\dot{\mathbf{q}}_0\|_M$. To begin with,

$$\left|\mathrm{fl}\left[M^{kj}\dot{\mathbf{q}}_0^j\right] - M^{kj}\dot{\mathbf{q}}_0^j\right| \leq \varepsilon\alpha_{14}$$

where

$$\alpha_{14} = 2\|M\|_\infty\|\dot{\mathbf{q}}_0\|_\infty + \|\dot{\mathbf{q}}_0\|_\infty + 1.$$

Since $\left|M^{kj}\dot{\mathbf{q}}_0^j\right| \leq d\|M\|_\infty\|\dot{\mathbf{q}}_0\|_\infty$, applying the summation formula yields

$$\left|\mathrm{fl}\left[(M\dot{\mathbf{q}}_0)^j\right] - (M\dot{\mathbf{q}}_0)^j\right| \leq \varepsilon\alpha_{15}$$

for

$$\alpha_{15} = d\alpha_{14} + (1 + d\|M\|_\infty\|\dot{\mathbf{q}}_0\|_\infty + \varepsilon\alpha_{14})\, d(1+\varepsilon)^{d-1}$$

Then

$$\left|\mathrm{fl}\left[\dot{\mathbf{q}}_0^j(M\dot{\mathbf{q}}_0)^j\right] - \dot{\mathbf{q}}_0^j(M\dot{\mathbf{q}}_0)^j\right| \leq \varepsilon\alpha_{16}$$

for

$$\alpha_{16} = d\|M\|_\infty\|\dot{\mathbf{q}}_0\|_\infty^2 + \|\dot{\mathbf{q}}_0^j\|\alpha_{15} + 1.$$

Applying the summation formula a second time gives

$$\left|\mathrm{fl}\left[\|\dot{\mathbf{q}}_0\|_M^2\right] - \|\dot{\mathbf{q}}_0\|_M^2\right| \leq \varepsilon\alpha_{17}$$

for

$$\alpha_{17} = d\alpha_{16} + \left(1 + \|\dot{\mathbf{q}}_0\|_M^2 + \varepsilon\alpha_{16}\right) d(1+\varepsilon)^{d-1}.$$

Finally

$$|\mathrm{fl}\left[\|\dot{\mathbf{q}}_0\|_M\right] - \|\dot{\mathbf{q}}_0\|_M| \leq \varepsilon\alpha_{18} \tag{6}$$

with

$$\alpha_{18} = 1 + \|\bar{\mathbf{q}}_0\|_M + \frac{1+\varepsilon}{\sqrt{\varepsilon}}\sqrt{\alpha_{17}}.$$

Combining equations (4) and (6) gives

$$\left|\mathrm{fl}\left[\|\dot{\mathbf{q}}_0\|_M\bar{p}^j\right] - \|\dot{\mathbf{q}}_0\|_M p^j\right| \leq \varepsilon(\alpha_{19} + \beta_{19}\sqrt{C})$$

where

$$\alpha_{19} = 1 + \frac{\|\dot{\mathbf{q}}_0\|_M^2}{\sqrt{\lambda_{\min}}} + 2\alpha_{18}\frac{\|\dot{\mathbf{q}}_0\|_M}{\sqrt{\lambda_{\min}}} + 2\|\dot{\mathbf{q}}_0\|\alpha_8 + 2\alpha_{18}\alpha_8$$

$$\beta_{19} = \|\dot{\mathbf{q}}_0\|_M\sqrt{\frac{2}{\lambda_{\min}}} + 2\alpha_{18}\sqrt{\frac{2}{\lambda_{\min}}} + 2\|\dot{\mathbf{q}}_0\|_M\beta_8 + 2\alpha_{18}\beta_8.$$

Now, we are at last prepared to bound the next velocity iterate

$$\dot{\mathbf{q}}_{i+1}^j = \mathrm{fl}\left[\frac{\|\dot{\mathbf{q}}_0\|_M\bar{\mathbf{p}}^j}{\|\bar{\mathbf{p}}\|_M}\right].$$

Suppose that $\|\mathbf{p}\|_M > \varepsilon(\alpha_{13} + \beta_{13}\sqrt{C})$. Then by the previous bound, and equation (5),

$$\left|\dot{\mathbf{q}}_{i+1}^j - \frac{\|\dot{\mathbf{q}}_0\|_M\mathbf{p}^j}{\|\mathbf{p}\|_M}\right| \leq \varepsilon\frac{\alpha_{20} + \beta_{20}\sqrt{C}}{\|\mathbf{p}\|_M - \varepsilon(\alpha_{13} + \beta_{13}\sqrt{C})} \tag{7}$$

for

$$\alpha_{20} = \frac{\|\dot{\mathbf{q}}_0\|_M}{\sqrt{\lambda_{\min}}}\alpha_{13} + 2\alpha_{19} + \frac{\|\dot{\mathbf{q}}_0\|_M^2}{\sqrt{\lambda_{\min}}} + \|\dot{\mathbf{q}}_0\|_M$$

$$\beta_{20} = \frac{\|\dot{\mathbf{q}}_0\|_M}{\sqrt{\lambda_{\min}}}\beta_{13} + 2\beta_{19} + \frac{\|\dot{\mathbf{q}}_0\|_M\sqrt{2}}{\sqrt{\lambda_{\min}}} + \sqrt{2}.$$

Therefore

$$\begin{aligned}
&\left|\|\dot{\mathbf{q}}_{i+1}\|_M^2 - \|\dot{\mathbf{q}}_0\|_M^2\|\right| \\
&\leq 2\|\dot{\mathbf{q}}_0\|\lambda_{\max}\sqrt{d}\varepsilon\frac{\alpha_{20} + \beta_{20}\sqrt{C}}{\|\mathbf{p}\|_M - \varepsilon(\alpha_{13} + \beta_{13}\sqrt{C})} \\
&\quad + \varepsilon\lambda_{\max}d\left(\frac{\alpha_{20} + \beta_{20}\sqrt{C}}{\|\mathbf{p}\|_M - \varepsilon(\alpha_{13} + \beta_{13}\sqrt{C})}\right)^2.
\end{aligned}$$

Let

$$\alpha_{21} = 4\lambda_{\max}\sqrt{d}\alpha_{20} + \frac{1}{\|\dot{\mathbf{q}}_0\|_M^2}4\lambda_{\max}d\alpha_{20}^2$$

$$\beta_{21} = 4\lambda_{\max}\sqrt{d}\beta_{20} + \frac{1}{\|\dot{\mathbf{q}}_0\|_M^2}8\lambda_{\max}d\alpha_{20}\beta_{20}$$

$$\gamma_{21} = \frac{1}{\|\dot{\mathbf{q}}_0\|_M^2}4\lambda_{\max}d\beta_{20}^2.$$

Lemma A.1. *If $\varepsilon < \frac{\|\dot{\mathbf{q}}_0\|_M}{2\alpha_{13}}$, $\varepsilon < \frac{2}{\gamma_{21}}$, and*

$$\frac{\varepsilon\beta_{21} + \sqrt{\varepsilon^2\beta_{21}^2 + 4\varepsilon\alpha_{21}(2 - \varepsilon\gamma_{21})}}{4 - 2\varepsilon\gamma_{21}} \leq \frac{(\|\dot{\mathbf{q}}_0\|_M - 2\varepsilon\alpha_{13})^2}{4(\sqrt{2} + \varepsilon\beta_{13})^2},$$

*then pairwise Gauss-Seidel satisfies ($\epsilon$DRIFT). Notice that these conditions are satisfied if $\varepsilon$ is sufficiently small.*

Proof. Take

$$C = \frac{1}{2}\left(\frac{(\|\dot{\mathbf{q}}_0\|_M - 2\varepsilon\alpha_{13})^2}{4(\sqrt{2} + \varepsilon\beta_{13})^2} + \frac{\varepsilon\beta_{21} + \sqrt{\varepsilon^2\beta_{21}^2 + 4\varepsilon\alpha_{21}(2 - \varepsilon\gamma_{21})}}{4 - 2\varepsilon\gamma_{21}}\right).$$

Since $\varepsilon < \frac{\|\dot{\mathbf{q}}_0\|_M}{2\alpha_{13}}$,

$$\|\dot{\mathbf{q}}_0\|_M - \sqrt{2C} - \varepsilon(\alpha_{13} + \beta_{13}\sqrt{C}) \geq \frac{1}{2}\|\dot{\mathbf{q}}_0\|_M$$

and

$$\|\mathbf{p}\|_M - \varepsilon(\alpha_{13} + \beta_{13}\sqrt{C}) \geq \frac{1}{2}\|\dot{\mathbf{q}}_0\|_M,$$

hence the bound in equation (7) is valid. Moreover we can substitute this inequality into the bound on $\|\dot{\mathbf{q}}_{i+1}\|_M^2$ to get

$$\left|\|\dot{\mathbf{q}}_{i+1}\|_M^2 - \|\dot{\mathbf{q}}_0\|_M^2\|\right| \leq \varepsilon(\alpha_{21} + \beta_{21}\sqrt{C} + \gamma_{21}C).$$

Then $\dot{\mathbf{q}}_{i+1}$ satisfies ($\epsilon$DRIFT) whenever

$$(2 - \varepsilon\gamma_{21})C - \varepsilon\beta_{21} - \varepsilon\alpha_{21} \leq 0,$$

and in particular, whenever

$$C \geq \frac{\varepsilon\beta_{21} + \sqrt{\varepsilon^2\beta_{21}^2 + 4\varepsilon\alpha_{21}(2 - \varepsilon\gamma_{21})}}{4 - 2\varepsilon\gamma_{21}}.$$

□

We now prove the remaining properties, which are relatively straightforward. First, we have that

Lemma A.2. *Let $C$ be as in the previous lemma, and suppose*

$$\epsilon > \sqrt{\varepsilon \frac{\lambda_{\max}\sqrt{d}(\alpha_{20} + \beta_{20}\sqrt{C})}{\|\dot{\mathbf{q}}_0\|(\|\dot{\mathbf{q}}_0\| + \sqrt{2C})}}$$

*and*

$$\epsilon \geq 2\varepsilon\lambda_{\max}\sqrt{d}\frac{2\alpha_{20} + 2\beta_{20}\sqrt{C}}{\|\dot{\mathbf{q}}_0\|_M(\|\dot{\mathbf{q}}_{0M} - \sqrt{2C})}.$$

*Then pairwise Gauss-Seidel satisfies ($\epsilon$NORM). Notice that both right-hand sides vanish as $\varepsilon$ decreases.*

Proof. Let $C$ be as in the previous lemma. By construction of the algorithm and ($\epsilon$VIO) we know that the value of $\lambda$ is

$$\lambda = -2\langle\dot{\mathbf{q}}_i, \mathbf{n}\rangle > 2\epsilon\|\dot{\mathbf{q}}_i\|_M \geq 2\epsilon(\|\dot{\mathbf{q}}_0\|_M + \sqrt{2C})$$

where the last inequality follows from ($\epsilon$DRIFT).

From the bound (7) on the components of $\mathbf{c}$, we have that

$$\|\mathbf{c}\|_M \leq \varepsilon\lambda_{\max}\sqrt{d}\frac{2\alpha_{20} + 2\beta_{20}\sqrt{C}}{\|\dot{\mathbf{q}}_0\|_M}$$

and this is less than $\epsilon\lambda$ when

$$\varepsilon\lambda_{\max}\sqrt{d}\frac{2\alpha_{20} + 2\beta_{20}\sqrt{C}}{\|\dot{\mathbf{q}}_0\|_M} \leq 2\epsilon^2(\|\dot{\mathbf{q}}_0\|_M + \sqrt{2C}).$$

Lastly since $\|\dot{\mathbf{q}}_i\|_M \geq \|\dot{\mathbf{q}}_0\|_M - \sqrt{2C}$, we have that $\|\mathbf{c}\|_M \leq \frac{\epsilon}{2}\|\dot{\mathbf{q}}_i\|$ whenever

$$\epsilon \geq 2\varepsilon\lambda_{\max}\sqrt{d}\frac{2\alpha_{20} + 2\beta_{20}\sqrt{C}}{\|\dot{\mathbf{q}}_0\|_M(\|\dot{\mathbf{q}}_{0M} - \sqrt{2C})}.$$

□

Lemma A.3. *Pairwise Gauss-Seidel satisfies ($\epsilon$MOD) when $\epsilon < 1$.*

Proof. At every iteration where a constraint with gradient $\mathbf{n}$ is violated,

$$\begin{aligned}\|\dot{\mathbf{q}}_{i+1} - \dot{\mathbf{q}}_i\|_M &= \|\lambda M^{-1}\mathbf{n} + \mathbf{c}\|_M \\ &\geq |\lambda| - \|\mathbf{c}\|_M \\ &\geq (1-\epsilon)|\lambda| \\ &> 0.\end{aligned}$$

□

## A.2 Generalized Reflections

The *generalized reflection* operator of Smith et al. [2012] improves on pairwise Gauss-Seidel by guaranteeing preservation of symmetries and more accurately modeling shock propagations, at the cost of an $R$ that is more expensive to compute. Algorithm 4 shows how to modify it so that it satisfies all the inexact desiderata required for guaranteed termination. Notice that these modifications mirror those of Gauss-Seidel: constraints whose violation does not exceed a threshold are pruned from consideration every time a reflection is applied, and the velocity is renormalized every step to prevent energy drift.

Computing $\lambda$ at each iteration of Algorithm 4 requires solving a quadratic program (QP). Let $\lambda$ be the exact solution to this QP, $\xi$ the corresponding positivity constraint Lagrange multipliers, and $\bar{\lambda}$, $\bar{\xi}$ the computed solution. We assume that $\bar{\lambda}$ approximately satisfies the KKT conditions of the QP,

$$\begin{aligned}\left\|N_V^T M^{-1} N_V\bar{\lambda} + 2N_V^T\dot{\mathbf{q}}_i - \bar{\xi}\right\|_\infty &\leq \varepsilon^2\kappa_1\|\dot{\mathbf{q}}_i\|_M \\ \bar{\lambda} &\geq 0 \\ \bar{\xi} &\geq 0 \\ \bar{\lambda} &\perp \bar{\xi},\end{aligned}$$

where $\kappa_1$ is an accuracy parameter independent of $\dot{\mathbf{q}}_i$; notice that this condition is a standard relative error termination criterion in numerical QP codes.

The goal now will be to bound the intermediate step

$$\bar{\mathbf{p}} = \mathrm{fl}\left[\dot{\mathbf{q}}_i + M^{-1}N_V\bar{\lambda}\right]$$

in terms of the true step $\mathbf{p} = \dot{\mathbf{q}}_i + M^{-1}N_V\lambda$; the proof of ($\epsilon$DRIFT) will then follow directly from identical calculations to that in pairwise Gauss-Seidel. Once we have a value of $C$, we will prove that inexact GR satisfies ($\epsilon$NORM) and ($\epsilon$MOD). As in the case of Gauss-Seidel, ($\epsilon$KIN), (ONE), and ($\epsilon$VIO) all hold by construction of Algortihm 4.

Let $N_A \subset N_V$ be the set of constraints that are active in the inexact QP solution, and $\bar{\lambda}_A$ the corresponding parts of $\bar{\lambda}$. The matrix $N_V^T M^{-1} N_V$ has ones along the diagonal, and off-diagonal entries of magnitude at most one; therefore by the Gershgorin Circle Theorem its maximum eigenvalue is at most $m$, the number of total constraints in $N$. Then we have the following useful bound on $\bar{\lambda}$:

$$\begin{aligned}\|\bar{\lambda}\|_\infty = \|\bar{\lambda}_A\|_\infty &\leq \frac{\|N_A^T M^{-1} N_A\bar{\lambda}_A\|_2}{m} \\ &\leq \frac{\sqrt{d}\|N_A^T M^{-1} N_A\bar{\lambda}_A\|_2}{m} \\ &\leq \frac{\sqrt{d}}{m}(\|2N_A^T\dot{\mathbf{q}}_i\|_\infty + \varepsilon\kappa_1\|\dot{\mathbf{q}}_i\|_M) \\ &\leq \frac{\sqrt{d}}{m}\left(\frac{2m}{\lambda_{\max}}\|\dot{\mathbf{q}}_i\|_M + \varepsilon\kappa_1\|\dot{\mathbf{q}}_i\|_M\right) \\ &\leq \kappa_2 + \mu_2\sqrt{C},\end{aligned}$$

**Algorithm 4** `Inexact Generalized Reflections`

**function** ResolveImpactsApprox($\mathbf{q}$, $\dot{\mathbf{q}}$, $\epsilon$)
  $N \leftarrow$ ActiveConstraintGradients($\mathbf{q}$)
  $\dot{\mathbf{q}}_0 \leftarrow \dot{\mathbf{q}}$
  **for** $i := 1, \infty$ **do**
    $N_V \leftarrow$ ViolatedN($\dot{\mathbf{q}}_i$)   // $\dot{\mathbf{q}}_i^T N_V < -\epsilon\|\dot{\mathbf{q}}_i\|_M\mathbf{1}$
    **if** $N_V = \emptyset$ **then**
      **return** $\dot{\mathbf{q}}_i$
    **end if**
    $\lambda \leftarrow \mathrm{argmin}_\lambda \|M^{-1}N_V\lambda + 2\dot{\mathbf{q}}_i\|_M^2$  s.t.  $\lambda \geq 0$
    $\dot{\mathbf{q}}_{i+1} \leftarrow \frac{\|\dot{\mathbf{q}}_0\|_M}{\|\dot{\mathbf{q}}_i + M^{-1}N_V\lambda\|_M}\left(\dot{\mathbf{q}}_i + M^{-1}N_V\lambda\right)$
  **end for**
**end function**

with

$$\kappa_2 = \left(\frac{2\sqrt{d}}{\lambda_{\max}} + \frac{\varepsilon\kappa_1}{m}\right)\|\dot{\mathbf{q}}_0\|_M$$
$$\mu_2 = \left(\frac{2\sqrt{d}}{\lambda_{\max}} + \frac{\varepsilon\kappa_1}{m}\right)\sqrt{2},$$

where as usual we have used $\epsilon^2 < \epsilon$ to simplify expressions.

Now for $\mathbf{n}_j$ the $j$th row of $N_V$, we have the bound

$$\left|\mathrm{fl}\left[\mathbf{n}_j^k\bar{\lambda}^k\right] - \mathbf{n}_j^k\bar{\lambda}^k\right| \le \varepsilon(\kappa_3 + \mu_3\sqrt{C})$$

where

$$\kappa_3 = 3\|N\|_\infty\kappa_2 + 2\kappa_2 + 1$$
$$\mu_3 = 3\|N\|_\infty\mu_2 + 2\mu_2,$$

so applying the summation formula gives

$$\left|\mathrm{fl}\left[\left(N_V\bar{\lambda}\right)^j\right] - \left(N_V\bar{\lambda}\right)^j\right| \le \varepsilon(\kappa_4 + \mu_4\sqrt{C})$$

for

$$\kappa_4 = (2\kappa_3 + 1 + m\|N\|_\infty\kappa_2)\,d(1+\varepsilon)^{d-1}$$
$$\mu_4 = (2\mu_3 + m\|N\|_\infty\mu_2)\,d(1+\varepsilon)^{d-1}.$$

Then

$$\left|\mathrm{fl}\left[\left(M^{-1}\right)^{kj}\left(N_V\bar{\lambda}\right)^j\right] - \left(M^{-1}\right)^{kj}\left(N_V\bar{\lambda}\right)^j\right|$$
$$\le \varepsilon(\kappa_5 + \mu_5\sqrt{C})$$

for

$$\kappa_5 = 1 + (5\|M^{-1}\|_\infty + 2)\kappa_4 + 2(\|M^{-1}\|_\infty + 1)\|N\|_\infty\kappa_2$$
$$\mu_5 = (5\|M^{-1}\|_\infty + 2)\mu_4 + 2(\|M^{-1}\|_\infty + 1)\|N\|_\infty\mu_2,$$

so that applying the summation formula gives

$$\left|\mathrm{fl}\left[\left(M^{-1}N_V\bar{\lambda}\right)^j\right] - \left(M^{-1}N_V\bar{\lambda}\right)^j\right| \le \varepsilon(\kappa_6 + \mu_6\sqrt{C})$$

for

$$\kappa_6 = d\kappa_5 + (1 + d\|M^{-1}\|_\infty\|N\|_\infty\kappa_2 + \kappa_5)d(1+\varepsilon)^{d-1}$$
$$\mu_6 = d\mu_5 + (\mu_5 + d\|M^{-1}\|_\infty\|N\|_\infty\mu_2)d(1+\varepsilon)^{d-1}.$$

Before we can bound $\bar{\mathbf{p}}$, we need to relate the impulse using the approximate multipliers $\bar{\lambda}$ to that using the exact multipliers. We can do so by making use of the fact that the QP's KKT conditions are nearly satisfied for $\bar{\lambda}$:

$$\begin{aligned}
&\|M^{-1}N_V\lambda - M^{-1}N_V\bar{\lambda}_V\|_M^2\\
&= (\lambda - \bar{\lambda})^T(N_V^TM^{-1}N_V\lambda - N_V^TM^{-1}N_V\bar{\lambda})\\
&\le -\langle\lambda,\bar{\xi}\rangle - \langle\bar{\lambda},\xi\rangle + \|\lambda - \bar{\lambda}\|_\infty\sqrt{d}\kappa_1\varepsilon^2\|\dot{\mathbf{q}}_i\|_M\\
&\le \|\lambda - \bar{\lambda}\|_\infty\sqrt{d}\kappa_1\varepsilon^2\|\dot{\mathbf{q}}_i\|_M\\
&\le (\|\lambda\|_\infty + \|\bar{\lambda}\|_\infty)\sqrt{d}\kappa_1\varepsilon^2(\|\dot{\mathbf{q}}_0\|_M + \sqrt{2C})\\
&\le 2(\kappa_2 + \mu_2\sqrt{C})\sqrt{d}\kappa_1\varepsilon^2(\|\dot{\mathbf{q}}_0\|_M + \sqrt{2C})\\
&\le \varepsilon^2(\kappa_7 + \mu_7\sqrt{C} + \nu_7C)
\end{aligned}$$

for

$$\kappa_7 = 2\|\dot{\mathbf{q}}_0\|_M\sqrt{d}\kappa_2\kappa_1$$
$$\mu_7 = 2\sqrt{2d}\kappa_2\kappa_1 + 2\|\dot{\mathbf{q}}_0\|_M\sqrt{d}\kappa_1\mu_2$$
$$\nu_7 = 2\sqrt{2d}\kappa_1\mu_2.$$

Completing the square gives

$$\begin{aligned}
&\|M^{-1}N_V\lambda - M^{-1}N_V\bar{\lambda}\|_\infty\\
&\le \frac{1}{\lambda_{\max}}\|M^{-1}N_V\lambda - M^{-1}N_V\bar{\lambda}\|_M\\
&\le \varepsilon(\kappa_8 + \mu_8\sqrt{C})
\end{aligned}$$

with

$$\kappa_8 = \frac{1}{\lambda_{\max}}\left(\frac{\mu_7}{2\sqrt{\nu_7}} + \sqrt{\left|\kappa_7 - \frac{\mu_7^2}{4\nu_7}\right|}\right)$$
$$\mu_8 = \frac{\sqrt{\nu_7}}{\lambda_{\max}}.$$

Therefore

$$\left|\mathrm{fl}\left[\left(M^{-1}N_V\bar{\lambda}\right)^j\right] - \left(M^{-1}N_V\lambda\right)^j\right| \le \varepsilon(\kappa_9 + \mu_9\sqrt{C})$$

where simply $\kappa_9 = \kappa_6 + \kappa_8$ and $\mu_9 = \mu_6 + \mu_8$. We then have

$$\left|\bar{\mathbf{p}}^j - \mathbf{p}^j\right| \le \varepsilon(\kappa_{10} + \mu_{10}\sqrt{C})$$

for

$$\kappa_{10} = 2\kappa_9 + d^2\|M^{-1}\|_\infty\|N_V\|_\infty\kappa_2 + \alpha_1 + 1$$
$$\mu_{10} = 2\mu_9 + d^2\|M^{-1}\|_\infty\|N_V\|_\infty\mu_2 + \beta_1.$$

The proof of ($\epsilon$DRIFT) now follows identically the arguments for pairwise Gauss-Seidel, with $\kappa_{10}$ and $\mu_{10}$ taking the place of $\alpha_8$ and $\beta_8$. As in the pairwise GS case, construction of a $C$ certifying ($\epsilon$DRIFT) requires that $\varepsilon$ be sufficiently small.

We now prove that GR satisfies the remaining properties, ($\epsilon$NORM) and ($\epsilon$MOD).

Lemma A.4. *Let $C$ be as in the proof of ($\epsilon$DRIFT), and suppose that*

$$\epsilon \ge \frac{a + \sqrt{a^2 + b}}{2},$$

*where*

$$a = \frac{\epsilon^2}{2}\kappa_1(\|\dot{\mathbf{q}}_0\|_M + \sqrt{2C})$$
$$b = 4m\varepsilon\lambda_{\max}\frac{\alpha_{20} + \beta_{20}\sqrt{C}}{\sqrt{d}\|\dot{\mathbf{q}}_0\|_M},$$

*and*

$$\epsilon \ge 2\varepsilon\lambda_{\max}\sqrt{d}\frac{2\alpha_{20} + 2\beta_{20}\sqrt{C}}{\|\dot{\mathbf{q}}_0\|_M(\|\dot{\mathbf{q}}_{0M} - \sqrt{2C})}.$$

*Then Generalized Reflections satisfies ($\epsilon$NORM). Notice that both right-hand sides vanish as $\varepsilon$ decreases.*

Proof. Since at least one constraint must be violated, by ($\epsilon$VIO),

$$\|N_V^TM^{-1}N_V\bar{\lambda}\|_2 \ge 2\epsilon\|\dot{\mathbf{q}}_i\|_M - \varepsilon^2\kappa_1\|\dot{\mathbf{q}}_i\|_M$$

and

$$\lambda \ge \frac{2\epsilon - \varepsilon^2\kappa_1}{m}(\|\dot{\mathbf{q}}_0\|_M + \sqrt{2C}),$$

where we have used ($\epsilon$DRIFT) and again the fact that the largest eigenvalue of $N_V^T M^{-1} N_V$ is at most $m$.

From the bound (7) on the components of $\mathbf{c}$, we have that

$$\|\mathbf{c}\|_M \leq \varepsilon \lambda_{\max} \sqrt{d} \frac{2\alpha_{20} + 2\beta_{20}\sqrt{C}}{\|\dot{\mathbf{q}}_0\|_M}$$

and this is less than $\epsilon \|\lambda\|_1$ when

$$\varepsilon \lambda_{\max} \sqrt{d} \frac{2\alpha_{20} + 2\beta_{20}\sqrt{C}}{\|\dot{\mathbf{q}}_0\|_M} \leq \epsilon d \frac{2\epsilon - \varepsilon^2 \kappa_1}{m} (\|\dot{\mathbf{q}}_0\|_M + \sqrt{2C}).$$

Rearranging gives

$$\epsilon^2 - \epsilon \frac{\varepsilon^2}{2} \kappa_1 (\|\dot{\mathbf{q}}_0\|_M + \sqrt{2C}) - m\varepsilon \lambda_{\max} \frac{\alpha_{20} + \beta_{20}\sqrt{C}}{\sqrt{d} \|\dot{\mathbf{q}}_0\|_M} \geq 0$$

and the first inequality above. Lastly since $\|\dot{\mathbf{q}}_i\|_M \geq \|\dot{\mathbf{q}}_0\|_M - \sqrt{2C}$, we have that $\|\mathbf{c}\|_M \leq \frac{\epsilon}{2} \|\dot{\mathbf{q}}_i\|$ whenever

$$\epsilon \geq 2\varepsilon \lambda_{\max} \sqrt{d} \frac{2\alpha_{20} + 2\beta_{20}\sqrt{C}}{\|\dot{\mathbf{q}}_0\|_M (\|\dot{\mathbf{q}}_{0M} - \sqrt{2C})},$$

as in the case of pairwise Gauss-Seidel. □

At last we end with

Lemma A.5. *If $\epsilon < 4$, then Generalized Reflections satisfies ($\epsilon$MOD).*

Proof. At every iteration where a constraint is violated,

$$\begin{aligned}
\|\dot{\mathbf{q}}_{i+1} - \dot{\mathbf{q}}_i\|_M &= \|M^{-1} N_V \lambda + \mathbf{c}\|_M \\
&\geq \|M^{-1} N_v \lambda\|_M - \|\mathbf{c}\|_M \\
&\geq \sqrt{2\|\lambda\|_1 \epsilon \|\dot{\mathbf{q}}_i\|_M} - \|\mathbf{c}\|_M \\
&\geq \sqrt{2\|\lambda\|_1 \epsilon \|\dot{\mathbf{q}}_i\|_M} - \epsilon \sqrt{\|\lambda\|_1 \|\dot{\mathbf{q}}_i\|_M / 2} \\
&\geq \sqrt{2\|\lambda\|_1 \epsilon \|\dot{\mathbf{q}}_i\|_M} - \epsilon \sqrt{\|\lambda\|_1 \|\dot{\mathbf{q}}_i\|_M / 2}.
\end{aligned}$$

The right-hand side is positive when $\epsilon < 4$. □