Supplementary Technical Document

Qi Guo, Xuchen Han, Chuyuan Fu, Theodore Gast, Rasmus Tamstorf, Joseph Teran

1 FEM Force computation

We compute forces on the control points $\mathbf{x}_p$ by

$$\begin{aligned}\mathbf{f}_p^{KL} &= -\frac{\partial \Psi^S(\mathbf{F}^{KL,Etr}(\mathbf{x}^{KL}))}{\partial \mathbf{x}_p^{KL}} \\ &= -\sum_q V_q^0 \frac{\partial \psi(\mathbf{F}_q^{KL,Etr}(\mathbf{x}_q))}{\partial \mathbf{x}_p^{KL}} \\ &= -\sum_q V_q^0 \frac{\partial \psi}{\partial \mathbf{F}^{KL}}(\mathbf{F}_q^{KL,Etr}(\mathbf{x}_q)) : \frac{\partial \mathbf{F}_q^{KL,Etr}}{\partial \mathbf{x}_p^{KL}}(\mathbf{x}_q),\end{aligned}$$

where $\mathbf{x}_q$'s are positions of the quadrature points. We give expressions for each $\frac{\partial \mathbf{F}_q^{KL}}{\partial x_{p_k}^{KL}}(\mathbf{x}_q)$ with fixed p, q and k, where k represents the x, y, or z direction. For simplicity of notation, we omit the subscripts p, q and superscript KL for now.

Recall from the paper that we have

$$\mathbf{F} = \sum_{i=1}^{3} \mathbf{g}_i \otimes \bar{\mathbf{g}}^i, \text{ with } \mathbf{g}_\alpha = \mathbf{a}_\alpha + \xi_3 \mathbf{a}_{3,\alpha}, \ \mathbf{g}_3 = \mathbf{a}_3,$$

where

$$\begin{aligned}\mathbf{a}_\alpha &= \sum_j \mathbf{x}_j \frac{\partial N_j^{SD}}{\partial \xi_\alpha}(\xi_1, \xi_2), \quad \alpha = 1, 2 \\ \mathbf{a}_3 &= \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} \\ \mathbf{a}_{3,\alpha} &= (\mathbf{I} - \mathbf{a}_3 \otimes \mathbf{a}_3) \frac{\mathbf{a}_{1,\alpha} \times \mathbf{a}_2 + \mathbf{a}_1 \times \mathbf{a}_{2,\alpha}}{|\mathbf{a}_1 \times \mathbf{a}_2|} \\ &= \tilde{\mathbf{a}} - \mathbf{a}_3(\mathbf{a}_3 \cdot \tilde{\mathbf{a}})\end{aligned}$$

in which we define $\tilde{\mathbf{a}}$ to be

$$\tilde{\mathbf{a}} = \frac{\mathbf{a}_{1,\alpha} \times \mathbf{a}_2 + \mathbf{a}_1 \times \mathbf{a}_{2,\alpha}}{|\mathbf{a}_1 \times \mathbf{a}_2|}.$$

Now we compute $\frac{\partial \mathbf{F}}{\partial x_k}$.

$$\frac{\partial \mathbf{F}}{\partial x_k} = \sum_{i=1}^{3} \frac{\partial \mathbf{g}_i}{\partial x_k} \otimes \bar{\mathbf{g}}^i,$$

and

$$\frac{\partial \mathbf{g}_\alpha}{\partial x_k} = \frac{\partial \mathbf{a}_\alpha}{\partial x_k} + \xi_3 \frac{\partial \mathbf{a}_{3,\alpha}}{\partial x_k}$$
$$\frac{\partial \mathbf{g}_3}{\partial x_k} = \frac{\partial \mathbf{a}_3}{\partial x_k}$$

where

$$\frac{\partial \mathbf{a}_\alpha}{\partial x_k} = \frac{\partial N_k^{SD}(\xi_1, \xi_2)}{\partial \xi_\alpha} \mathbf{e}_k \text{ (summation convention does not apply here)} \quad (1)$$

$$\frac{\partial \mathbf{a}_3}{\partial x_k} = \frac{\frac{\partial \mathbf{a}_1}{\partial x_k} \times \mathbf{a}_2 + \mathbf{a}_1 \times \frac{\partial \mathbf{a}_2}{\partial x_k} - \frac{|\mathbf{a}_1 \times \mathbf{a}_2|}{\partial x_k} \mathbf{a}_3}{|\mathbf{a}_1 \times \mathbf{a}_2|},$$

and

$$\frac{|\mathbf{a}_1 \times \mathbf{a}_2|}{\partial x_k} = \mathbf{a}_3 \cdot (\frac{\partial \mathbf{a}_1}{\partial x_k} \times \mathbf{a}_2 + \mathbf{a}_1 \times \frac{\partial \mathbf{a}_2}{\partial x_k})$$

Finally,

$$\frac{\partial \mathbf{a}_{3,\alpha}}{\partial x_k} = \frac{\partial \tilde{\mathbf{a}}}{\partial x_k} - \mathbf{a}_3 (\frac{\partial \mathbf{a}_3}{\partial x_k} \cdot \tilde{\mathbf{a}} + \mathbf{a}_3 \cdot \frac{\partial \tilde{\mathbf{a}}}{\partial x_k}) - \frac{\partial \mathbf{a}_3}{\partial x_k} (\mathbf{a}_3 \cdot \tilde{\mathbf{a}}),$$

where

$$\frac{\partial \tilde{\mathbf{a}}}{\partial x_k} = \frac{\frac{\mathbf{a}_{1,\alpha}}{\partial x_k} \times \mathbf{a}_2 + \mathbf{a}_{1,\alpha} \times \frac{\partial \mathbf{a}_2}{\partial x_k} + \frac{\partial \mathbf{a}_1}{\partial x_k} \times \mathbf{a}_{2,\alpha} + \mathbf{a}_1 \times \frac{\partial \mathbf{a}_{2,\alpha}}{\partial x_k}}{|\mathbf{a}_1 \times \mathbf{a}_2|} - \frac{\mathbf{a}_{1,\alpha} \times \mathbf{a}_2 + \mathbf{a}_1 \times \mathbf{a}_{2,\alpha}}{|\mathbf{a}_1 \times \mathbf{a}_2|^2} \frac{\partial |\mathbf{a}_1 \times \mathbf{a}_2|}{\partial x_k},$$

in which

$$\frac{\partial \mathbf{a}_{\alpha,\beta}}{\partial x_k} = \frac{N_k^{SD}(\xi_1, \xi_2)}{\partial \xi_\beta \partial \xi_\alpha} \mathbf{e}_k \text{ (summation convention does not apply here).}$$

2 Grid force computation

The force on the MPM grid $f_{\mathbf{i}}^{iii}(\mathbf{x}^*)$ computes as follows:

$$
\begin{aligned}
\mathbf{f}_{\mathbf{i}}^{(iii)}(\mathbf{x}^*) &= \sum_{p\in\mathcal{I}^{(iii)}} \frac{\partial\chi(\mathbf{a}_{p\alpha}\otimes\bar{\mathbf{a}}_{p\alpha}+\mathbf{a}_{p3}^E\otimes\bar{\mathbf{a}}_{p3})}{\partial\mathbf{x}_{\mathbf{i}}}V_p^0 \\
&= \sum_{p\in\mathcal{I}^{(iii)}} \frac{\partial\chi(\mathbf{a}_{p\alpha}\otimes\bar{\mathbf{a}}_{p\alpha}+\mathbf{a}_{p3}^E\otimes\bar{\mathbf{a}}_{p3})}{\partial\mathbf{F}^E} : \frac{\partial\left(\mathbf{a}_{p\alpha}\otimes\bar{\mathbf{a}}_{p\alpha}+\mathbf{a}_{p3}^E\otimes\bar{\mathbf{a}}_{p3}\right)}{\partial\mathbf{a}_{p\beta}} : \frac{\partial\mathbf{a}_{p\beta}}{\mathbf{x}_{\mathbf{i}}}V_p^0 \\
&+ \sum_{p\in\mathcal{I}^{(iii)}} \frac{\partial\chi(\mathbf{a}_{p\alpha}\otimes\bar{\mathbf{a}}_{p\alpha}+\mathbf{a}_{p3}^E\otimes\bar{\mathbf{a}}_{p3})}{\partial\mathbf{F}^E} : \frac{\partial\left(\mathbf{a}_{p\alpha}\otimes\bar{\mathbf{a}}_{p\alpha}+\mathbf{a}_{p3}^E\otimes\bar{\mathbf{a}}_{p3}\right)}{\partial\mathbf{a}_{p3}^E} : \frac{\partial\mathbf{a}_{p3}^E}{\mathbf{x}_{\mathbf{i}}}V_p^0.
\end{aligned}
$$

Then, omitting the subscript p, we compute each term in the contraction:

$$
\begin{aligned}
\frac{\partial\chi(\mathbf{a}_{\alpha}\otimes\bar{\mathbf{a}}_{\alpha}+\mathbf{a}_{3}^E\otimes\bar{\mathbf{a}}_{3})}{\partial\mathbf{F}^E} &= \boldsymbol{\tau}^S\left(\mathbf{a}_{\alpha}\otimes\bar{\mathbf{a}}_{\alpha}+\mathbf{a}_{3}^E\otimes\bar{\mathbf{a}}_{3})\right)^{-T} \\
&= \boldsymbol{\tau}^S\left(\tilde{\mathbf{a}}^{\alpha}\otimes\bar{\mathbf{a}}^{\alpha}+\tilde{\mathbf{a}}^{3}\otimes\bar{\mathbf{a}}_{3})\right)
\end{aligned}
$$

where $\boldsymbol{\tau}^S$ is the Kirchhoff stress and $\tilde{\mathbf{a}}^{\alpha}$ and $\tilde{\mathbf{a}}^{3}$ are the contravariant counterparts of $\mathbf{a}_\alpha$ and $\mathbf{a}_3^E$ respectively.

And using index notation, we see that

$$
\begin{aligned}
\frac{\partial\left(\mathbf{a}_{\alpha}\otimes\bar{\mathbf{a}}_{\alpha}+\mathbf{a}_{3}^E\otimes\bar{\mathbf{a}}_{3}\right)}{\partial\mathbf{a}_{\beta}} &= \frac{\partial a_{\alpha_i}\bar{a}_{\alpha_j}}{\partial a_{\beta_k}} \\
&= \delta_{\alpha\beta}\delta_{ik}\bar{a}_{\alpha_j} \\
&= \delta_{ik}\bar{a}_{\beta_j}
\end{aligned}
$$

Similarly,

$$
\frac{\partial\left(\mathbf{a}_{\alpha}\otimes\bar{\mathbf{a}}_{\alpha}+\mathbf{a}_{3}^E\otimes\bar{\mathbf{a}}_{3}\right)}{\partial\mathbf{a}_{3}^E} = \delta_{ik}\bar{a}_{3_j}
$$

Hence, contracting the first two terms in the summation, each term in the summation becomes

$$
\begin{aligned}
&\boldsymbol{\tau}^S\left(\tilde{\mathbf{a}}^{\alpha}\otimes\bar{\mathbf{a}}^{\alpha}+\tilde{\mathbf{a}}_{3}\otimes\bar{\mathbf{a}}_{3}\right)\bar{\mathbf{a}}_{\beta} : \frac{\partial\mathbf{a}_{\beta}}{\partial\mathbf{x}_{\mathbf{i}}} + \boldsymbol{\tau}^S\left(\tilde{\mathbf{a}}^{\alpha}\otimes\bar{\mathbf{a}}^{\alpha}+\tilde{\mathbf{a}}_{3}\otimes\bar{\mathbf{a}}_{3}\right)\bar{\mathbf{a}}_{3} : \frac{\partial\mathbf{a}_{3}^E}{\partial\mathbf{x}_{\mathbf{i}}} \\
&= \boldsymbol{\tau}^S\tilde{\mathbf{a}}^{\beta} : \frac{\partial\mathbf{a}_{\beta}}{\partial\mathbf{x}_{\mathbf{i}}} + \boldsymbol{\tau}^S\tilde{\mathbf{a}}^{3} : \frac{\partial\mathbf{a}_{3}^E}{\partial\mathbf{x}_{\mathbf{i}}}
\end{aligned}
$$

Note that

$$\frac{\partial \mathbf{a}_\beta}{\partial \mathbf{x_i}} = \frac{\partial \mathbf{a}_\beta}{\partial \mathbf{x}_p}\frac{\partial \mathbf{x}_p}{\partial \mathbf{x_i}} = \frac{\partial \mathbf{a}_\beta}{\partial \mathbf{x}_p} w_{\mathbf{i}p}^n,$$

and the expression for $\frac{\partial \mathbf{a}_\beta}{\partial \mathbf{x}_p}$ is given equation (1).

Ignoring further plastic flow, we have

$$\mathbf{a}_3^E(\mathbf{x}^*) = \left(\sum_{\mathbf{j}} \mathbf{x}_{\mathbf{j}}^* \otimes \nabla w_{\mathbf{j}p}^n\right) \mathbf{a}_3^{E,n},$$

and thus,

$$\frac{\partial \mathbf{a}_3^E}{\partial \mathbf{x_i}} = \nabla w_{\mathbf{i}p}^n \mathbf{a}_3^{E,n}$$

Therefore, we arrive at the final expression for the force of type (iii):

$$\mathbf{f}_{\mathbf{i}}^{(iii)}(\mathbf{x}^*) = \sum_{p \in \mathcal{I}^{(iii)}} \boldsymbol{\tau}_p^S \tilde{\mathbf{a}}_p^\beta : \frac{\partial \mathbf{a}_{p\beta}}{\partial \mathbf{x}_p} w_{\mathbf{i}p}^n + \boldsymbol{\tau}_p^S \tilde{\mathbf{a}}_p^3 : \nabla w_{\mathbf{i}p}^n \mathbf{a}_{p3}^{E,n}$$

3 Laminate Stress

In this section we derive the expression for

$$\boldsymbol{\tau}^{KL} = \tau_{\alpha\beta} \mathbf{q}_\alpha^{KL,E} \otimes \mathbf{q}_\beta^{KL,E}, \quad \tau_{\alpha\beta}^{KL} = 2\mu\epsilon_{\alpha\beta}^L + \lambda\epsilon_{\gamma\gamma}^L \delta_{\alpha\beta}. \tag{2}$$

First notice that we may replace the right Hencky strain with left Hencky strain in the definition of energy because of the isotropic nature of the energy function. We now give the drivation of Equation (2) with index free notation assuming all variables are in 2D.

$$\begin{aligned} \psi(\mathbf{F}) &= \psi(\mathbf{U\Sigma V}^T) \\ \mathbf{P}(\mathbf{F}) &= \mathbf{P}(\mathbf{U\Sigma V}^T) = \mathbf{U}\mathbf{P}(\mathbf{\Sigma})\mathbf{V}^T \end{aligned}$$

because the energy is isotropic.

Hence,

$$\begin{aligned} \mathbf{P}(\mathbf{F}) &= \mathbf{U}\mathbf{P}(\mathbf{\Sigma})\mathbf{V}^T \\ &= \mathbf{U}\frac{\partial \psi}{\partial \mathbf{\Sigma}}\mathbf{V}^T \\ &= \mathbf{U}\left(2\mu \log(\mathbf{\Sigma})\mathbf{\Sigma}^{-1} + \lambda \mathrm{tr}(\log \mathbf{\Sigma})\mathbf{\Sigma}^{-1}\right)\mathbf{V}^T. \end{aligned}$$

Therefore,

$$\begin{aligned}\boldsymbol{\tau}^{KL} &= \left(\mathbf{U}\left(2\mu\log(\boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1} + \lambda\mathrm{tr}(\log\boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1}\right)\mathbf{V}^T\right)\mathbf{F}^T \\ &= \mathbf{U}\left(2\mu\log(\boldsymbol{\Sigma}) + \lambda\mathrm{tr}(\log\boldsymbol{\Sigma})\right)\mathbf{U}^T \\ &= 2\mu\boldsymbol{\epsilon}^L + \lambda\mathrm{tr}(\boldsymbol{\epsilon}^L)\end{aligned}$$

4 QR and Elastic Potential

We can use QR orthogonalization of deformed material directions to define

$$\mathbf{q}_i r_{ij} = \mathbf{F}\bar{\mathbf{a}}_j,\ \mathbf{F} = r_{ij}\mathbf{q}_i \otimes \bar{\mathbf{a}}_j,\ r_{ij} = 0 \text{ for } i > j. \tag{3}$$

4.1 Change of basis tensor

Define the change of basis tensor

$$\mathbf{Q} = Q_{ij}\bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_j \tag{4}$$

with $Q_{ij} = \mathbf{q}_j \cdot \bar{\mathbf{a}}_i$. With this convention we see that $\mathbf{Q}\bar{\mathbf{a}}_i = \mathbf{q}_i$ and $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. Furthermore, defining

$$\mathbf{R} = r_{ij}\bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_j$$

we have $\mathbf{F} = \mathbf{QR}$.

4.2 Differentials

The QR differential satisfies

$$\mathbf{q}_k \cdot \delta\mathbf{q}_i r_{ij} + \delta r_{kj} = \mathbf{q}_k \cdot (\delta\mathbf{F}\bar{\mathbf{a}}_j),\ \delta\mathbf{F} = \delta r_{ij}\mathbf{q}_i \otimes \bar{\mathbf{a}}_j + r_{ij}\delta\mathbf{q}_i \otimes \bar{\mathbf{a}}_j \tag{5}$$

where $\mathbf{q}_k \cdot \delta\mathbf{q}_i = -\mathbf{q}_i \cdot \delta\mathbf{q}_k$ from orthogonality of the $\mathbf{q}_i$. And

$$\delta\mathbf{F} = \delta\mathbf{Q}\mathbf{R} + \mathbf{Q}\delta\mathbf{R} \tag{6}$$

where $\delta\mathbf{Q}^T\mathbf{Q} = -\mathbf{Q}^T\delta\mathbf{Q}$ from $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. Furthermore,

$$\delta\mathbf{Q} = \delta Q_{ij}\bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_j,\ \delta Q_{ij} = \delta\mathbf{q}_j \cdot \bar{\mathbf{a}}_i,\ \delta\mathbf{q}_i = \delta\mathbf{Q}\bar{\mathbf{a}}_i \tag{7}$$

$$\delta\mathbf{R} = \delta r_{ij}\bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_j \tag{8}$$

and the $\delta r_{ij} = 0$ for $i > j$.

5 Elastic potential and stresses

Define the hyperelastic potential as

$$\psi(\mathbf{F}) = \hat{\psi}([\mathbf{R}]) \tag{9}$$

where

$$[\mathbf{R}] = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{pmatrix}. \tag{10}$$

The differential satisfies

$$\delta\psi(\mathbf{F}) = \frac{\partial\psi}{\partial\mathbf{F}}(\mathbf{F}) : \delta\mathbf{F} = \mathbf{P} : \delta\mathbf{F} = \frac{\partial\hat{\psi}}{\partial r_{ij}}([\mathbf{R}])\delta r_{ij} \tag{11}$$

where $\mathbf{P} = \frac{\partial\psi}{\partial\mathbf{F}}(\mathbf{F})$. Therefore

$$\delta r_{ij}\mathbf{q}_i \cdot (\mathbf{P}\bar{\mathbf{a}}_j) + r_{ij}\delta\mathbf{q}_i \cdot (\mathbf{P}\bar{\mathbf{a}}_j) = \frac{\partial\hat{\psi}}{\partial r_{ij}}([\mathbf{R}])\delta r_{ij}. \tag{12}$$

Similarly,

$$\mathbf{P} : \delta\mathbf{F} = \mathbf{P} : (\delta\mathbf{Q}\mathbf{R}) + \mathbf{P} : (\mathbf{Q}\delta\mathbf{R}) = \frac{\partial\hat{\psi}}{\partial r_{ij}}([\mathbf{R}])\delta r_{ij} \tag{13}$$

Choosing $\delta\mathbf{F} = \delta r_{ij}\mathbf{q}_i \otimes \bar{\mathbf{a}}_j$ (i.e. $\delta\mathbf{q}_i = \mathbf{0}$), we can conclude that

$$\mathbf{q}_i \cdot (\mathbf{P}\bar{\mathbf{a}}_j)\,\delta r_{ij} = \frac{\partial\hat{\psi}}{\partial r_{ij}}([\mathbf{R}])\delta r_{ij} \tag{14}$$

for arbitrary δr_{ij} with $i \leq j$. Therefore the $\mathbf{q}_i \cdot (\mathbf{P}\bar{\mathbf{a}}_j) = \frac{\partial\hat{\psi}}{\partial r_{ij}}([\mathbf{R}])$ for $i \leq j$. Similarly,

$$\mathbf{P} : (\mathbf{Q}\delta\mathbf{R}) = (\mathbf{Q}^T\mathbf{P}) : \delta\mathbf{R} = \delta r_{ij}\bar{\mathbf{a}}_i \cdot \left(\mathbf{Q}^T\mathbf{P}\bar{\mathbf{a}}_j\right) = \delta r_{ij}\mathbf{q}_i \cdot (\mathbf{P}\bar{\mathbf{a}}_j) = \frac{\partial\hat{\psi}}{\partial r_{ij}}([\mathbf{R}])\delta r_{ij}. \tag{15}$$

Choosing $\delta\mathbf{F} = r_{ij}\delta\mathbf{q}_i \otimes \bar{\mathbf{a}}_j$ (i.e. $\delta r_{ij} = 0$), we can conclude that

$$0 = r_{ij}\delta\mathbf{q}_i \cdot (\mathbf{P}\bar{\mathbf{a}}_j)\,. \tag{16}$$

Similarly,

$$0 = \mathbf{P} : (\delta\mathbf{Q}\mathbf{R}) = \left(\mathbf{P}\mathbf{R}^T\right) : \delta\mathbf{Q} = \left(\mathbf{P}\mathbf{R}^T\right) : \left(\delta\mathbf{Q}\mathbf{Q}^T\mathbf{Q}\right) = \left(\mathbf{P}\mathbf{R}^T\mathbf{Q}^T\right) : \left(\delta\mathbf{Q}\mathbf{Q}^T\right) = \left(\mathbf{P}\mathbf{F}^T\right) : \left(\delta\mathbf{Q}\mathbf{Q}^T\right) \tag{17}$$

In other words, the Kirchhoff stress $\boldsymbol{\tau} = \mathbf{P}\mathbf{F}^T$ is symmetric since $\delta\mathbf{Q}\mathbf{Q}^T$ is arbitrary skew. Furthermore,

$$\mathbf{P} = P_{ij}\mathbf{q}_i \otimes \bar{\mathbf{a}}_j, \;\; \boldsymbol{\tau} = P_{ij}r_{kj}\mathbf{q}_i \otimes \mathbf{q}_k = \tau_{ik}\mathbf{q}_i \otimes \mathbf{q}_k \tag{18}$$

and we know $P_{ij} = \frac{\partial\hat{\psi}}{\partial r_{ij}}$ for $i \le j$ from Equation 14. Thus

$$\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \begin{pmatrix} r_{11} & & \\ r_{12} & r_{22} & \\ r_{13} & r_{23} & r_{33} \end{pmatrix} \tag{19}$$

$$= \begin{pmatrix} P_{11}r_{11} + P_{12}r_{12} + P_{13}r_{13} & P_{12}r_{22} + P_{13}r_{32} & P_{13}r_{33} \\ P_{21}r_{11} + P_{22}r_{12} + P_{23}r_{13} & P_{22}r_{22} + P_{23}r_{32} & P_{23}r_{33} \\ P_{31}r_{11} + P_{32}r_{12} + P_{33}r_{13} & P_{32}r_{22} + P_{33}r_{32} & P_{33}r_{33} \end{pmatrix}, \tag{20}$$

and since $\boldsymbol{\tau} = \boldsymbol{\tau}^T$ and $P_{ij} = \frac{\partial\hat{\psi}}{\partial r_{ij}}$ for $i \le j$,

$$\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial\hat{\psi}}{\partial r_{11}}r_{11} + \frac{\partial\hat{\psi}}{\partial r_{12}}r_{12} + \frac{\partial\hat{\psi}}{\partial r_{13}}r_{13} & \frac{\partial\hat{\psi}}{\partial r_{12}}r_{22} + \frac{\partial\hat{\psi}}{\partial r_{13}}r_{32} & \frac{\partial\hat{\psi}}{\partial r_{13}}r_{33} \\ \frac{\partial\hat{\psi}}{\partial r_{12}}r_{22} + \frac{\partial\hat{\psi}}{\partial r_{13}}r_{32} & \frac{\partial\hat{\psi}}{\partial r_{22}}r_{22} + \frac{\partial\hat{\psi}}{\partial r_{23}}r_{32} & \frac{\partial\hat{\psi}}{\partial r_{23}}r_{33} \\ \frac{\partial\hat{\psi}}{\partial r_{13}}r_{33} & \frac{\partial\hat{\psi}}{\partial r_{23}}r_{33} & \frac{\partial\hat{\psi}}{\partial r_{33}}r_{33} \end{pmatrix} \tag{21}$$

In particular, the matrix representation of $\boldsymbol{\tau}^S$ reads

$$\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \gamma s_1 \\ 0 & 0 & \gamma s_2 \\ 0 & 0 & f'(s_3) \end{pmatrix} \begin{pmatrix} 0 & & \\ 0 & 0 & \\ s_1 & s_2 & s_3 \end{pmatrix} \tag{22}$$

$$= \begin{pmatrix} \gamma s_1^2 & \gamma s_1 s_2 & \gamma s_1 s_3 \\ \gamma s_1 s_2 & \gamma s_2^2 & \gamma s_2 s_3 \\ \gamma s_1 s_3 & \gamma s_2 s_3 & f'(s_3) \end{pmatrix} \tag{23}$$

6 Frictional Contact Yield Condition

Coulomb friction places a constraint on the stress as

$$|\mathbf{t}_S| \leq -c_F \sigma_n \tag{24}$$

where $\sigma_n = \mathbf{a}_3^{KL} \cdot \boldsymbol{\sigma} \mathbf{a}_3^{KL}$. Recall that $\mathbf{a}_3^{KL} = \mathbf{q}_3$ and thus $\sigma_n = \mathbf{q}_3 \cdot \boldsymbol{\sigma} \mathbf{q}_3$. On the other hand, $\mathbf{t}_S$ is the tangential component of the force density and has the form $\mathbf{t}_S = (c\mathbf{q}_1 + s\mathbf{q}_2) \cdot \boldsymbol{\sigma} \mathbf{q}_3$ for some c and s such that $c^2 + s^2 = 1$. Hence, we may rewrite the constraint on stress as

$$(c\mathbf{q}_1 + s\mathbf{q}_2) \cdot \boldsymbol{\sigma} \mathbf{q}_3 + c_F \mathbf{q}_3 \cdot \boldsymbol{\sigma} \mathbf{q}_3 \leq 0. \tag{25}$$

Using the fact that $\boldsymbol{\sigma} = \det(\mathbf{F})\boldsymbol{\tau}$, we rewrite the constraint as

$$(c\mathbf{q}_1 + s\mathbf{q}_2) \cdot \boldsymbol{\tau} \mathbf{q}_3 + c_F \mathbf{q}_3 \cdot \boldsymbol{\tau} \mathbf{q}_3 \leq 0. \tag{26}$$

Substituting in the expression for $\boldsymbol{\tau}$ from equation (23), we find that the maximum on the left-hand-side is

$$\pm \gamma s_3 \sqrt{s_1^2 + s_2^2} + c_F f' s_3$$

We apply the particular form of f in the paper where $f(x) = \frac{1}{3} k^c (1-x)^3$ for $x \leq 1$ and 0 otherwise. When $s_3 > 1$, the maximum is $\gamma s_3 \sqrt{s_1^2 + s_2^2}$. In this case the return mapping set s_1 and s_2 to 0. If $0 < s_3 \leq 1$, the maximum is

$$\gamma s_3 \sqrt{s_1^2 + s_2^2} - c_F k^c (s_3 - 1)^2 s_3,$$

and thus we need

$$\sqrt{s_1^2 + s_2^2} \leq \frac{c_F k^c}{\gamma} (1 - s_3)^2.$$

In this case we uniformly scale back s_1 and s_2 to satisfy the constraint.

7 Denting Yield Condition and Return Mapping

We apply the von Mises yield condition to the Kirchhoff-Stress in Equation (2)

This condition states that the deviatoric component of the stress is less than a threshold value c_{vM}

$$f_{vM}(\boldsymbol{\tau}) = |\boldsymbol{\tau} - \frac{\operatorname{tr}(\boldsymbol{\tau})}{3} \mathbf{I}|_F \leq c_{vM}. \tag{27}$$

This condition defines a cylindrical region of feasible states in the principal stress space since

$$f_{vM}(\boldsymbol{\tau}) = \sqrt{\frac{2}{3}\left({\tau_1}^2 + {\tau_2}^2 + {\tau_3}^2 - (\tau_1\tau_2 + \tau_2\tau_3 + \tau_1\tau_3)\right)} \tag{28}$$

where $\boldsymbol{\tau} = \sum_i \tau_i \mathbf{u}_i \otimes \mathbf{u}_i$ with principal stresses τ_i. The plane stress nature of $\boldsymbol{\tau}^{KL} = \sum_\alpha \tau_\alpha^{KL} \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha$ means that feasible stresses are those where the principal stresses are in the ellipsoidal intersection of the cylinder and the τ_α^{KL} plane.

The yield condition is satisfied via associative projection (or return mapping) of the stress to the feasible region. The elastic and plastic strains are then computed to be consistent with the projected stress. We use $\mathbf{F}^{KL,E^{\mathrm{tr}}}, \mathbf{F}^{KL,P^{\mathrm{tr}}}$ to denote the trial state of elastoplastic strains with associated trial stress $\boldsymbol{\tau}^{KL^{\mathrm{tr}}}$. We use $\mathbf{F}^{KL,E}, \mathbf{F}^{KL,P}, \boldsymbol{\tau}^{KL}$ to denote their projected counterparts.

$$\mathbf{F}^{KL,E^{\mathrm{tr}}}, \mathbf{F}^{KL,P^{\mathrm{tr}}}, \boldsymbol{\tau}^{KL^{\mathrm{tr}}} \rightarrow \mathbf{F}^{KL,E}, \mathbf{F}^{KL,P}, \boldsymbol{\tau}^{KL}. \tag{29}$$

The deformation gradient constraint must be equal to the product of trial and projected elastic and plastic deformation gradients, creating the constraint on the projection

$$\mathbf{F}^{KL} = \mathbf{F}^{KL,E^{\mathrm{tr}}}\mathbf{F}^{KL,P^{\mathrm{tr}}} = \mathbf{F}^{KL,E}\mathbf{F}^{KL,P}. \tag{30}$$

The projection is completed by first computing the trial state of stress $\boldsymbol{\tau}^{KL^{\mathrm{tr}}}$ from $\mathbf{F}^{KL,E^{\mathrm{tr}}}$ using Equation (2). This is done by computing the QR decomposition of the trial elastic deformation gradient $\mathbf{F}^{KL,E^{\mathrm{tr}}} = r_{\alpha\beta}^{KL,E^{\mathrm{tr}}} \mathbf{q}_\alpha^{KL,E} \otimes \bar{\mathbf{a}}_\beta + \mathbf{q}_3^{KL,E} \otimes \bar{\mathbf{a}}_3$. Then we compute the SVD of matrix $[\mathbf{r}^{KL,E^{\mathrm{tr}}}] \in \mathbb{R}^{2\times 2}$ and the trial strain $[\boldsymbol{\epsilon}^{L^{\mathrm{tr}}}]$

$$[\mathbf{r}^{KL,E^{\mathrm{tr}}}] = [\mathbf{U}^E] \begin{pmatrix} \sigma_1^{E^{\mathrm{tr}}} & \\ & \sigma_2^{E^{\mathrm{tr}}} \end{pmatrix} [\mathbf{V}^E]^T \tag{31}$$

$$[\boldsymbol{\epsilon}^{L^{\mathrm{tr}}}] = [\mathbf{U}^E] \begin{pmatrix} \log(\sigma_1^{E^{\mathrm{tr}}}) & \\ & \log(\sigma_2^{E^{\mathrm{tr}}}) \end{pmatrix} [\mathbf{U}^E]^T \tag{32}$$

From Equation (2) we see that the two non-zero principal stresses ${\tau^{KL}}_\alpha^{\mathrm{tr}}$ of $\boldsymbol{\tau}^{KL^{\mathrm{tr}}}$ are equal to the eigenvalues of the matrix $[\boldsymbol{\tau}^{KL^{\mathrm{tr}}}]$

$$[\boldsymbol{\tau}^{KL^{\mathrm{tr}}}] = 2\mu[\boldsymbol{\epsilon}^{L^{\mathrm{tr}}}] + \lambda \mathrm{tr}([\boldsymbol{\epsilon}^{L^{\mathrm{tr}}}])\mathbf{I} = [\mathbf{U}^E] \begin{pmatrix} \tau_1^{KL^{\mathrm{tr}}} & \\ & \tau_2^{KL^{\mathrm{tr}}} \end{pmatrix} [\mathbf{U}^E]^T. \tag{33}$$

We therefore project the eigenvalues (${\tau^{KL}}^{\text{tr}}_{\alpha} \rightarrow {\tau^{KL}}_{\alpha}$) into the ellipsoidal intersection the von Mises yield surface and the (τ_1, τ_2) plane in the direction that maximizes energy dissipation. We approximate this region by the diamond shaped region whose boundaries have slopes of ± 1 to simplify the return mapping. Note that the direction of the return that maximizes energy dissipation is a function of the Cauchy-Green strain derivative of the Kirchhoff stress and thus is non-trivial to find in general. Fortunately, the quadratic Hencky strain model has the favorable property that the return direction is perpendicular to the yield surface [1] which greatly simplifies the return mapping. We illustrate this property in Figure 1. After projection, we rebuild the matrix without changing the eigenvectors and rebuild $\boldsymbol{\tau}^{KL}$ from the matrix

$$[\boldsymbol{\tau}^{KL}] = [\mathbf{U}^E] \begin{pmatrix} {\tau^{KL}}_1 & \\ & {\tau^{KL}}_2 \end{pmatrix} [\mathbf{U}^E]^T, \ \boldsymbol{\tau}^{KL} = \tau^{KL}_{\alpha\beta} \mathbf{q}^{KL,E}_{\alpha} \otimes \mathbf{q}^{KL,E}_{\beta} \tag{34}$$

where $\tau^{KL}_{\alpha\beta}$ are the entries in the projected matrix $[\boldsymbol{\tau}^{KL}] \in \mathbb{R}^{2\times 2}$. The projected strain $[\boldsymbol{\epsilon}^L]$ is computed from the projected principal stresses from

$$[\boldsymbol{\epsilon}^L] = [\mathbf{U}^E] \begin{pmatrix} \log(\sigma^E_1) & \\ & \log(\sigma^E_2) \end{pmatrix} [\mathbf{U}^E]^T \tag{35}$$

$$\begin{pmatrix} \log(\sigma^E_1) \\ \log(\sigma^E_2) \end{pmatrix} = \begin{pmatrix} 2\mu + \lambda & \lambda \\ \lambda & 2\mu + \lambda \end{pmatrix}^{-1} \begin{pmatrix} {\tau^{KL}}_1 \\ {\tau^{KL}}_2 \end{pmatrix} \tag{36}$$

and the projected elastic deformation gradient is $\mathbf{F}^{KL,E} = F^{KL,E}_{\alpha\beta} \mathbf{q}^{KL,E}_{\alpha} \otimes \bar{\mathbf{a}}_{\beta} + \mathbf{q}^{KL,E}_{3} \otimes \bar{\mathbf{a}}_3$ where

$$[\hat{\mathbf{F}}^{KL,E}] = [\mathbf{U}^E] \begin{pmatrix} \sigma^E_1 & \\ & \sigma^E_2 \end{pmatrix} [\mathbf{V}^E]^T. \tag{37}$$

The projected plastic deformation gradient is computed from $\mathbf{F}^{KL,P} = {\mathbf{F}^{KL,E}}^{-1} \mathbf{F}^{KL}$ in order to maintain the constraint in Equation (30).

References

[1] C. Mast. *Modeling landslide-induced flow interactions with structures using the Material Point Method.* PhD thesis, 2013.

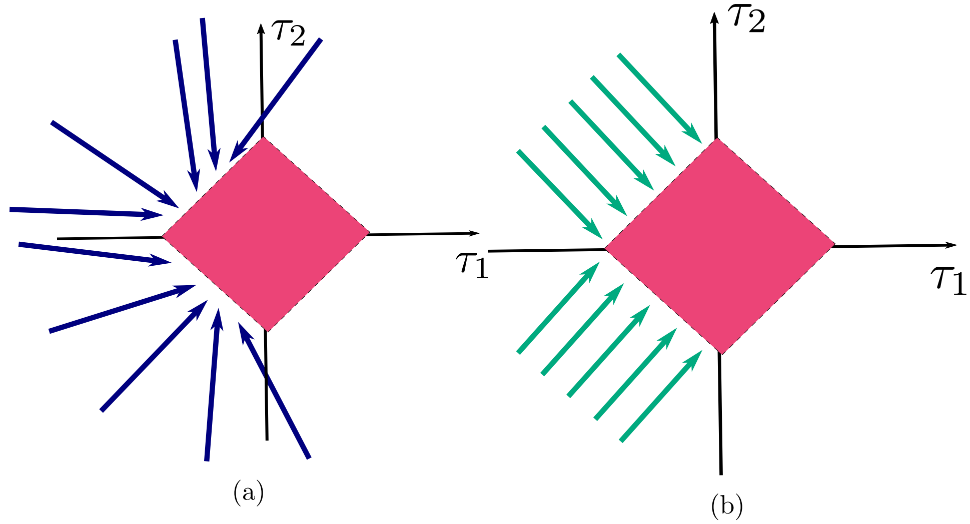

Figure 1: **Return Mapping**. In general in the return mapping direction is non trivial (left). Quadratic Hencky strain energy density simplifies the return mapping (right).